Associative Memories as a Building Block in Transformers

Alberto Bietti

Flatiron Institute, Simons Foundation

Overparametrization, Regularization, Identifiability and Uncertainty in Machine Learning.
Oberwolfach, January 2025

SIMONS FOUNDATION FLATIRON INSTITUTE

Associative Memories as a Building Block in Transformers

Alberto Bietti

Flatiron Institute, Simons Foundation

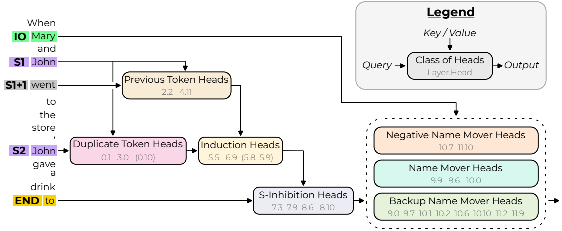
Overparametrization, Regularization, Identifiability and Uncertainty in Machine Learning.
Oberwolfach, January 2025

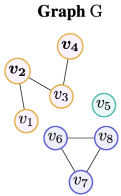
w/ V. Cabannes, E. Dohmatob, D. Bouchacourt, H. Jégou, L. Bottou (Meta AI),
E. Nichani, J. Lee (Princeton), B. Simsek, L. Chen, J. Bruna (NYU)

What are Transformer LLMs doing?

Reasoning over context

- Circuits of attention heads (Elhage et al., 2021; Olsson et al., 2022; Wang et al., 2022)
- Many results on expressivity (e.g., circuits, formal languages, graph connectivity)
  - e.g., (Merrill et al., 2022; Liu et al., 2023; Sanford et al., 2023)





Task: Are v_2 and v_4 connected?

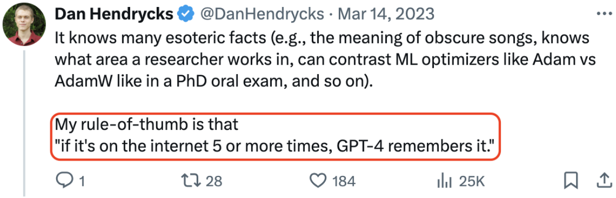
What are Transformer LLMs doing?

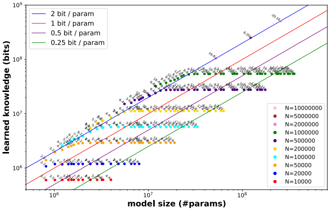
Reasoning over context

- Circuits of attention heads (Elhage et al., 2021; Olsson et al., 2022; Wang et al., 2022)
- Many results on expressivity (e.g., circuits, formal languages, graph connectivity)
  - e.g., (Merrill et al., 2022; Liu et al., 2023; Sanford et al., 2023)

Knowledge storage

- Memorization, factual recall, parameter scaling
  - (Geva et al., 2020; Meng et al., 2022; Allen-Zhu and Li, 2024)
- Allows higher-level reasoning





What are Transformer LLMs doing?

Reasoning over context

- Circuits of attention heads (Elhage et al., 2021; Olsson et al., 2022; Wang et al., 2022)
- Many results on expressivity (e.g., circuits, formal languages, graph connectivity)
  - e.g., (Merrill et al., 2022; Liu et al., 2023; Sanford et al., 2023)

Knowledge storage

- Memorization, factual recall, parameter scaling
  - (Geva et al., 2020; Meng et al., 2022; Allen-Zhu and Li, 2024)
- Allows higher-level reasoning

Goal: tractable model for both + training dynamics?

Transformer setup

Input: sequence of discrete tokens $(z_1, \ldots, z_T) \in [N]^T$

Transformer setup

Input: sequence of discrete tokens $(z_1, \dots, z_T) \in [N]^T$

Embeddings

- input e_z, positional p_t, output u_y, in $\mathbb{R}^d$
- this talk: **fixed** to **random** init $\mathcal{N}(0, 1/d)$

Transformer setup

Input: sequence of discrete tokens $(z_1, \ldots, z_T) \in [N]^T$

Embeddings

- input e_z, positional p_t, output u_y, in $\mathbb{R}^d$
- this talk: **fixed** to **random** init $\mathcal{N}(0, 1/d)$

Residual streams (Elhage et al., 2021)

- embed each token $z_t \in [N]$ as $x_t := e_{z_t} + p_t$
- (causal) self-attention $x_t := x_t + \mathrm{MHSA}(x_t, x_{1:t})$

$$\mathrm{MHSA}(x_t, x_{1:t}) = \sum_{h=1}^{H} \sum_{s=1}^{t} \beta_s^h W_O^{h\top} W_V^h x_s, \quad \text{with } \beta_s^h = \frac{\exp(x_s^\top W_K^{h\top} W_Q^h x_t)}{\sum_{s=1}^{t} \exp(x_s^\top W_K^{h\top} W_Q^h x_t)}$$

where $W_K, W_Q, W_V, W_O \in \mathbb{R}^{d_h \times d}$ (key/query/value/output matrices)

Transformer setup

Input: sequence of discrete tokens $(z_1, \dots, z_T) \in [N]^T$

Embeddings

- input e_z, positional p_t, output u_y, in $\mathbb{R}^d$
- this talk: **fixed** to **random** init $\mathcal{N}(0, 1/d)$

Residual streams (Elhage et al., 2021)

- embed each token $z_t \in [N]$ as $x_t := e_{z_t} + p_t$
- (causal) self-attention $x_t := x_t + \mathrm{MHSA}(x_t, x_{1:t})$
- feed-forward $x_t := x_t + \mathrm{MLP}(x_t)$

$$\mathrm{MLP}(x_t) = V^\top \sigma(U x_t)$$

where $U, V \in \mathbb{R}^{m \times d}$, often $m = 4d$

Transformer setup

Input: sequence of discrete tokens $(z_1, \ldots, z_T) \in [N]^T$

Embeddings

- input e_z, positional p_t, output u_y, in $\mathbb{R}^d$
- this talk: **fixed** to **random** init $\mathcal{N}(0, 1/d)$

Residual streams (Elhage et al., 2021)

- embed each token $z_t \in [N]$ as $x_t := e_{z_t} + p_t$
- (causal) self-attention $x_t := x_t + \text{MHSA}(x_t, x_{1:t})$
- feed-forward $x_t := x_t + \text{MLP}(x_t)$
- residual stream x_t is a sum of embeddings/"features"

Transformer setup

Input: sequence of discrete tokens $(z_1, \ldots, z_T) \in [N]^T$

Embeddings

- input e_z, positional p_t, output u_y, in $\mathbb{R}^d$
- this talk: **fixed** to **random** init $\mathcal{N}(0, 1/d)$

Residual streams (Elhage et al., 2021)

- embed each token $z_t \in [N]$ as $x_t := e_{z_t} + p_t$
- (causal) self-attention $x_t := x_t + \mathrm{MHSA}(x_t, x_{1:t})$
- feed-forward $x_t := x_t + \mathrm{MLP}(x_t)$
- residual stream x_t is a sum of embeddings/"features"

Next-token prediction

- cross-entropy loss

$$\sum_{t<T} \ell(z_{t+1}; (u_j{}^\top x_t)_j)$$

Outline

1. Associative memories
2. Application to Transformers I: reasoning (B. et al., 2023)
3. Application to Transformers II: factual recall (Nichani et al., 2024)

Weights as associative memories

- Consider sets of **nearly orthonormal embeddings** $\{e_z\}_{z \in \mathcal{Z}}$ and $\{u_y\}_{y \in \mathcal{Y}}$:

$$\|e_z\| \approx 1 \quad \text{and} \quad {e_z}^\top e_{z'} \approx 0$$
$$\|u_y\| \approx 1 \quad \text{and} \quad {u_y}^\top u_{y'} \approx 0$$

Weights as associative memories

- Consider sets of **nearly orthonormal embeddings** $\{e_z\}_{z \in \mathcal{Z}}$ and $\{u_y\}_{y \in \mathcal{Y}}$:

$$\|e_z\| \approx 1 \quad \text{and} \quad {e_z}^\top e_{z'} \approx 0$$
$$\|u_y\| \approx 1 \quad \text{and} \quad {u_y}^\top u_{y'} \approx 0$$

- Consider **pairwise associations** $(z, y) \in \mathcal{M}$ with **weights** α_{zy} and define:

$$W = \sum_{(z,y) \in \mathcal{M}} \alpha_{zy} u_y {e_z}^\top$$

Weights as associative memories

- Consider sets of **nearly orthonormal embeddings** $\{e_z\}_{z\in\mathcal{Z}}$ and $\{u_y\}_{y\in\mathcal{Y}}$:

$$\|e_z\| \approx 1 \quad \text{and} \quad {e_z}^\top e_{z'} \approx 0$$
$$\|u_y\| \approx 1 \quad \text{and} \quad {u_y}^\top u_{y'} \approx 0$$

- Consider **pairwise associations** $(z, y) \in \mathcal{M}$ with **weights** α_{zy} and define:

$$W = \sum_{(z,y)\in\mathcal{M}} \alpha_{zy} u_y {e_z}^\top \quad \Longrightarrow \quad {u_y}^\top W e_z \approx \alpha_{zy}$$

Weights as associative memories

- Consider sets of **nearly orthonormal embeddings** $\{e_z\}_{z \in \mathcal{Z}}$ and $\{u_y\}_{y \in \mathcal{Y}}$:

$$\|e_z\| \approx 1 \quad \text{and} \quad {e_z}^\top e_{z'} \approx 0$$
$$\|u_y\| \approx 1 \quad \text{and} \quad {u_y}^\top u_{y'} \approx 0$$

- Consider **pairwise associations** $(z, y) \in \mathcal{M}$ with **weights** α_{zy} and define:

$$W = \sum_{(z,y) \in \mathcal{M}} \alpha_{zy} u_y {e_z}^\top \quad \Longrightarrow \quad {u_y}^\top W e_z \approx \alpha_{zy}$$

- Examples in Transformers:
  - Logits in attention heads: $x_k^\top W_{KQ} x_q$
  - Logits in next-token prediction: $u_y^\top U \sigma(V x_t)$ or $u_y^\top W_{OV} x_k$

Weights as associative memories

- Consider sets of **nearly orthonormal embeddings** $\{e_z\}_{z\in\mathcal{Z}}$ and $\{u_y\}_{y\in\mathcal{Y}}$:

$$\|e_z\| \approx 1 \quad \text{and} \quad {e_z}^\top e_{z'} \approx 0$$
$$\|u_y\| \approx 1 \quad \text{and} \quad {u_y}^\top u_{y'} \approx 0$$

- Consider **pairwise associations** $(z, y) \in \mathcal{M}$ with **weights** α_{zy} and define:

$$W = \sum_{(z,y)\in\mathcal{M}} \alpha_{zy} u_y {e_z}^\top \quad \Longrightarrow \quad {u_y}^\top W e_z \approx \alpha_{zy}$$

- Examples in Transformers:
  - Logits in attention heads: $x_k^\top W_{KQ} x_q$
  - Logits in next-token prediction: $u_y^\top U\sigma(Vx_t)$ or $u_y^\top W_{OV} x_k$
- Related to Hopfield (1982); Kohonen (1972); Willshaw et al. (1969); Iscen et al. (2017)

Weights as associative memories

- Consider sets of **nearly orthonormal embeddings** $\{e_z\}_{z \in \mathcal{Z}}$ and $\{u_y\}_{y \in \mathcal{Y}}$:

$$\|e_z\| \approx 1 \quad \text{and} \quad {e_z}^\top e_{z'} \approx 0$$
$$\|u_y\| \approx 1 \quad \text{and} \quad {u_y}^\top u_{y'} \approx 0$$

- Consider **pairwise associations** $(z, y) \in \mathcal{M}$ with **weights** α_{zy} and define:

$$W = \sum_{(z,y) \in \mathcal{M}} \alpha_{zy} u_y {e_z}^\top \quad \implies \quad {u_y}^\top W e_z \approx \alpha_{zy}$$

- Examples in Transformers:
  - Logits in attention heads: $x_k^\top W_{KQ} x_q$
  - Logits in next-token prediction: $u_y^\top U \sigma(V x_t)$ or $u_y^\top W_{OV} x_k$
- Related to Hopfield (1982); Kohonen (1972); Willshaw et al. (1969); Iscen et al. (2017)
- Note: attention itself is also related to AM (Ramsauer et al., 2020; Schlag et al., 2021)

Gradient associative memories

Lemma (Gradients as memories, B. et al., 2023)

Let p be a data distribution over $(z, y) \in [N]^2$, and consider the loss

$$L(W) = \mathbb{E}_{(z,y)\sim p}[\ell(y, F_W(z))], \quad F_W(z)_k = u_k{}^\top W e_z,$$

with ℓ the ***cross-entropy loss*** *and e_z, u_k input/output embeddings.*

Gradient associative memories

Lemma (Gradients as memories, B. et al., 2023)

Let p be a data distribution over $(z, y) \in [N]^2$, and consider the loss

$$L(W) = \mathbb{E}_{(z,y)\sim p}[\ell(y, F_W(z))], \quad F_W(z)_k = u_k{}^\top W e_z,$$

*with ℓ the **cross-entropy loss** and e_z, u_k input/output embeddings. Then,*

$$\nabla L(W) = \sum_{k=1}^{K} \mathbb{E}_z[(\hat{p}_W(y = k|z) - p(y = k|z)) u_k e_z{}^\top]$$

Gradient associative memories

Lemma (Gradients as memories, B. et al., 2023)

Let p be a data distribution over $(z, y) \in [N]^2$, and consider the loss

$$L(W) = \mathbb{E}_{(z,y)\sim p}[\ell(y, F_W(z))], \quad F_W(z)_k = {u_k}^\top W e_z,$$

*with ℓ the **cross-entropy loss** and e_z, u_k input/output embeddings. Then,*

$$\nabla L(W) = \sum_{k=1}^{K} \mathbb{E}_z[(\hat{p}_W(y = k|z) - p(y = k|z)) u_k {e_z}^\top]$$

- **Example**: $z \sim \mathrm{Unif}([N])$, $y = f_*(z)$

Gradient associative memories

Lemma (Gradients as memories, B. et al., 2023)

Let p be a data distribution over $(z, y) \in [N]^2$, and consider the loss

$$L(W) = \mathbb{E}_{(z,y)\sim p}[\ell(y, F_W(z))], \quad F_W(z)_k = {u_k}^\top W e_z,$$

*with ℓ the **cross-entropy loss** and e_z, u_k input/output embeddings. Then,*

$$\nabla L(W) = \sum_{k=1}^{K} \mathbb{E}_z[(\hat{p}_W(y = k|z) - p(y = k|z)) u_k {e_z}^\top]$$

- **Example**: $z \sim \mathrm{Unif}([N])$, $y = f_*(z)$
  - After **one gradient step** on the population loss, assuming near-orthonormal embeddings

$$W_1 = \frac{\eta}{N} \sum_{z,k} \left(\mathbb{1}\{f_*(z) = k\} - \frac{1}{N} \right) u_k {e_z}^\top \quad \Longrightarrow \quad {u_k}^\top W_1 e_z \approx \frac{\eta}{N} \left(\mathbb{1}\{f_*(z) = k\} - \frac{1}{N} \right)$$

Gradient associative memories

Lemma (Gradients as memories, B. et al., 2023)

Let p be a data distribution over $(z, y) \in [N]^2$, and consider the loss

$$L(W) = \mathbb{E}_{(z,y)\sim p}[\ell(y, F_W(z))], \quad F_W(z)_k = {u_k}^\top W e_z,$$

with ℓ the ***cross-entropy loss*** *and e_z, u_k input/output embeddings. Then,*

$$\nabla L(W) = \sum_{k=1}^{K} \mathbb{E}_z[(\hat{p}_W(y = k|z) - p(y = k|z)) u_k {e_z}^\top]$$

- **Example**: $z \sim \mathrm{Unif}([N])$, $y = f_*(z)$
  - After **one gradient step** on the population loss, assuming near-orthonormal embeddings

$$W_1 = \frac{\eta}{N} \sum_{z,k} \left(\mathbb{1}\{f_*(z) = k\} - \frac{1}{N} \right) u_k {e_z}^\top \quad \Longrightarrow \quad {u_k}^\top W_1 e_z \approx \frac{\eta}{N} \left(\mathbb{1}\{f_*(z) = k\} - \frac{1}{N} \right)$$

  - **Corollary**: $\hat{f}(z) = \arg\max_k {u_k}^\top W_1 e_z$ has near-perfect accuracy

Gradient associative memories

Lemma (Gradients as memories, B. et al., 2023)

Let p be a data distribution over $(z, y) \in [N]^2$, and consider the loss

$$L(W) = \mathbb{E}_{(z,y)\sim p}[\ell(y, F_W(z))], \quad F_W(z)_k = u_k{}^\top W e_z,$$

*with ℓ the **cross-entropy loss** and e_z, u_k input/output embeddings. Then,*

$$\nabla L(W) = \sum_{k=1}^{K} \mathbb{E}_z[(\hat{p}_W(y = k|z) - p(y = k|z)) u_k e_z{}^\top]$$

- **Example**: $z \sim \mathrm{Unif}([N])$, $y = f_*(z)$
  - After **one gradient step** on the population loss, assuming near-orthonormal embeddings

$$W_1 = \frac{\eta}{N} \sum_{z,k} \left(\mathbb{1}\{f_*(z) = k\} - \frac{1}{N} \right) u_k e_z{}^\top \quad \Longrightarrow \quad u_k{}^\top W_1 e_z \approx \frac{\eta}{N} \left(\mathbb{1}\{f_*(z) = k\} - \frac{1}{N} \right)$$

  - **Corollary**: $\hat{f}(z) = \arg\max_k u_k{}^\top W_1 e_z$ has near-perfect accuracy
- More generally, replace u_k by "backward" vector

Gradient associative memories

Lemma (Gradients as memories, B. et al., 2023)

Let p be a data distribution over $(z, y) \in [N]^2$, and consider the loss

$$L(W) = \mathbb{E}_{(z,y)\sim p}[\ell(y, F_W(z))], \quad F_W(z)_k = u_k{}^\top W e_z,$$

*with ℓ the **cross-entropy loss** and e_z, u_k input/output embeddings. Then,*

$$\nabla L(W) = \sum_{k=1}^{K} \mathbb{E}_z[(\hat{p}_W(y = k|z) - p(y = k|z)) u_k e_z{}^\top]$$

- **Example**: $z \sim \text{Unif}([N])$, $y = f_*(z)$
  - After **one gradient step** on the population loss, assuming near-orthonormal embeddings

$$W_1 = \frac{\eta}{N} \sum_{z,k} \left(\mathbb{1}\{f_*(z) = k\} - \frac{1}{N} \right) u_k e_z{}^\top \quad \Longrightarrow \quad u_k{}^\top W_1 e_z \approx \frac{\eta}{N} \left(\mathbb{1}\{f_*(z) = k\} - \frac{1}{N} \right)$$

  - **Corollary**: $\hat{f}(z) = \arg\max_k u_k{}^\top W_1 e_z$ has near-perfect accuracy
- More generally, replace u_k by "backward" vector

Note: related to (Ba et al., 2022; Damian et al., 2022; Oymak et al., 2023; Yang and Hu, 2021)

Capacity: Intuition

- Random embeddings $e_z, u_y \sim \mathcal{N}(0, \frac{1}{d}I)$

Capacity: Intuition

- Random embeddings $e_z, u_y \sim \mathcal{N}(0, \frac{1}{d} I)$
- For some $f^* : [N] \to [M]$

$$W = \sum_{z=1}^{N} u_{f^*(z)} e_z{}^\top \in \mathbb{R}^{d \times d}$$

Capacity: Intuition

- Random embeddings $e_z, u_y \sim \mathcal{N}(0, \frac{1}{d} I)$
- For some $f^* : [N] \to [M]$

$$W = \sum_{z=1}^{N} u_{f^*(z)} e_z{}^\top \in \mathbb{R}^{d \times d}$$

- When can we recover $\arg\max_y \gamma_{z,y} = f^*(z)$ for all z?

$$\gamma_{z,y} := u_y{}^\top W e_z = \sum_{z'} u_y^\top u_{f^*(z')} e_z^\top e_{z'}$$

Capacity: Intuition

- Random embeddings $e_z, u_y \sim \mathcal{N}(0, \frac{1}{d} I)$
- For some $f^* : [N] \to [M]$

$$W = \sum_{z=1}^{N} u_{f^*(z)} e_z^\top \in \mathbb{R}^{d \times d}$$

- When can we recover $\arg\max_y \gamma_{z,y} = f^*(z)$ for all z?

$$\gamma_{z,y} := u_y^\top W e_z = \sum_{z'} u_y^\top u_{f^*(z')} e_z^\top e_{z'}$$

$$\mathbb{E}[\gamma_{z,y}] = \begin{cases} 1, & \text{if } y = f^*(z) \\ 0, & \text{otherwise.} \end{cases} \qquad \mathrm{Var}[\gamma_{z,y}] \lesssim \frac{|\{f^*(z') = y\}|}{d} + \frac{|\{f^*(z') \neq y\}|}{d^2} \overset{?}{\lesssim} 1$$

Capacity: Intuition

- Random embeddings $e_z, u_y \sim \mathcal{N}(0, \frac{1}{d} I)$
- For some $f^* : [N] \to [M]$

$$W = \sum_{z=1}^{N} u_{f^*(z)} e_z^\top \in \mathbb{R}^{d \times d}$$

- When can we recover $\arg\max_y \gamma_{z,y} = f^*(z)$ for all z?

$$\gamma_{z,y} := u_y^\top W e_z = \sum_{z'} u_y^\top u_{f^*(z')} e_z^\top e_{z'}$$

$$\mathbb{E}[\gamma_{z,y}] = \begin{cases} 1, & \text{if } y = f^*(z) \\ 0, & \text{otherwise.} \end{cases} \qquad \mathrm{Var}[\gamma_{z,y}] \lesssim \frac{|\{f^*(z') = y\}|}{d} + \frac{|\{f^*(z') \neq y\}|}{d^2} \overset{?}{\lesssim} 1$$

- **Examples**: (Cabannes, Dohmatob, and B., 2024a; Nichani, Lee, and B., 2024)
  - f^* injective: can store up to $N \approx d^2$ associations (much better than one hot!)

Capacity: Intuition

- Random embeddings $e_z, u_y \sim \mathcal{N}(0, \frac{1}{d} I)$
- For some $f^* : [N] \to [M]$

$$W = \sum_{z=1}^{N} u_{f^*(z)} e_z^\top \in \mathbb{R}^{d \times d}$$

- When can we recover $\arg\max_y \gamma_{z,y} = f^*(z)$ for all z?

$$\gamma_{z,y} := u_y^\top W e_z = \sum_{z'} u_y^\top u_{f^*(z')} e_z^\top e_{z'}$$

$$\mathbb{E}[\gamma_{z,y}] = \begin{cases} 1, & \text{if } y = f^*(z) \\ 0, & \text{otherwise.} \end{cases} \qquad \mathrm{Var}[\gamma_{z,y}] \lesssim \frac{|\{f^*(z') = y\}|}{d} + \frac{|\{f^*(z') \neq y\}|}{d^2} \overset{?}{\lesssim} 1$$

- **Examples**: (Cabannes, Dohmatob, and B., 2024a; Nichani, Lee, and B., 2024)
  - f^* injective: can store up to $N \approx d^2$ associations (much better than one hot!)
  - $f^*(z) = z \mod 2$: can store up to $N \approx d$ associations

Capacity: Intuition

- Random embeddings $e_z, u_y \sim \mathcal{N}(0, \frac{1}{d} I)$
- For some $f^* : [N] \to [M]$

$$W = \sum_{z=1}^{N} u_{f^*(z)} e_z^\top \in \mathbb{R}^{d \times d}$$

- When can we recover $\arg\max_y \gamma_{z,y} = f^*(z)$ for all z?

$$\gamma_{z,y} := u_y^\top W e_z = \sum_{z'} u_y^\top u_{f^*(z')} e_z^\top e_{z'}$$

$$\mathbb{E}[\gamma_{z,y}] = \begin{cases} 1, & \text{if } y = f^*(z) \\ 0, & \text{otherwise.} \end{cases} \qquad \mathrm{Var}[\gamma_{z,y}] \lesssim \frac{|\{f^*(z') = y\}|}{d} + \frac{|\{f^*(z') \neq y\}|}{d^2} \overset{?}{\lesssim} 1$$

- **Examples**: (Cabannes, Dohmatob, and B., 2024a; Nichani, Lee, and B., 2024)
  - f^* injective: can store up to $N \approx d^2$ associations (much better than one hot!)
  - $f^*(z) = z \mod 2$: can store up to $N \approx d$ associations
  - Scaling laws: store the most frequent tokens with under-parameterized model

Capacity $\approx$ number of parameters

Low-rank

- $W = W_1^\top W_2$, with $W_1, W_2 \in \mathbb{R}^{m \times d}$ (e.g., key-query or output-value matrices)
- can store $N \approx md$ associations when $m \leq d$
- construction: random W_1, one step on W_2

(Nichani, Lee, and B., 2024), related to Krotov and Hopfield (2016); Demircigil et al. (2017)

Capacity $\approx$ number of parameters

Low-rank

- $W = W_1^\top W_2$, with $W_1, W_2 \in \mathbb{R}^{m \times d}$ (e.g., key-query or output-value matrices)
- can store $N \approx md$ associations when $m \leq d$
- construction: random W_1, one step on W_2

Non-linear MLP

- $\hat{f}(z) = \arg\max_y u_y{}^\top W_1 \sigma(W_2^\top e_z)$, $W_1, W_2 \in \mathbb{R}^{d \times m}$
- can store $N \approx md$ associations for any width m
- construction: using Hermite polynomials of degree $\approx \log N / \log d$ in kernel regime

(Nichani, Lee, and B., 2024), related to Krotov and Hopfield (2016); Demircigil et al. (2017)

Capacity ≈ number of parameters

Low-rank

- $W = W_1^\top W_2$, with $W_1, W_2 \in \mathbb{R}^{m \times d}$ (e.g., key-query or output-value matrices)
- can store $N \approx md$ associations when $m \leq d$
- construction: random W_1, one step on W_2

Non-linear MLP

- $\hat{f}(z) = \arg\max_y u_y{}^\top W_1 \sigma(W_2^\top e_z)$, $W_1, W_2 \in \mathbb{R}^{d \times m}$
- can store $N \approx md$ associations for any width m
- construction: using Hermite polynomials of degree $\approx \log N / \log d$ in kernel regime

Multi-input

- $\hat{f}(z_1, z_2) = \arg\max_y u_y{}^\top W_1 \sigma(W_2^\top (e_{z_1} + \tilde{e}_{z_2}))$
- also $N \approx md$ capacity

(Nichani, Lee, and B., 2024), related to Krotov and Hopfield (2016); Demircigil et al. (2017)

Capacity $\approx$ number of parameters

Low-rank

- $W = W_1^\top W_2$, with $W_1, W_2 \in \mathbb{R}^{m \times d}$ (e.g., key-query or output-value matrices)
- can store $N \approx md$ associations when $m \leq d$
- construction: random W_1, one step on W_2

Non-linear MLP

- $\hat{f}(z) = \arg\max_y u_y{}^\top W_1 \sigma(W_2^\top e_z)$, $W_1, W_2 \in \mathbb{R}^{d \times m}$
- can store $N \approx md$ associations for any width m
- construction: using Hermite polynomials of degree $\approx \log N / \log d$ in kernel regime

Multi-input

- $\hat{f}(z_1, z_2) = \arg\max_y u_y{}^\top W_1 \sigma(W_2^\top (e_{z_1} + \tilde{e}_{z_2}))$
- also $N \approx md$ capacity

Note: matches information-theoretic lower bounds

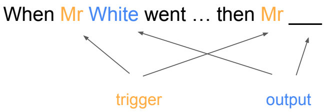
(Nichani, Lee, and B., 2024), related to Krotov and Hopfield (2016); Demircigil et al. (2017)

Outline

1. Associative memories
2. Application to Transformers I: reasoning (B. et al., 2023)
3. Application to Transformers II: factual recall (Nichani et al., 2024)

The bigram data model for in-context reasoning

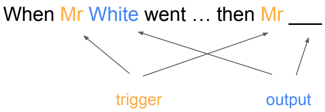
Goal: capture both in-context and global knowledge (*e.g.*, nouns vs syntax)



When Mr White went to the mall, it started raining, then Mr White witnessed an odd occurrence. While walking around the mall with his family, Mr White heard the sound of a helicopter landing in the parking lot. Curious, he made his way over to see what was going on.

The bigram data model for in-context reasoning

Goal: capture both in-context and global knowledge (*e.g.*, nouns vs syntax)

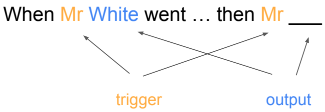


When Mr White went to the mall, it started raining, then Mr White witnessed an odd occurrence. While walking around the mall with his family, Mr White heard the sound of a helicopter landing in the parking lot. Curious, he made his way over to see what was going on.

Fix **trigger tokens**: $q_1, \ldots, q_K$

The bigram data model for in-context reasoning

Goal: capture both in-context and global knowledge (*e.g.*, nouns vs syntax)



When Mr White went to the mall, it started raining, then Mr White witnessed an odd occurrence. While walking around the mall with his family, Mr White heard the sound of a helicopter landing in the parking lot. Curious, he made his way over to see what was going on.

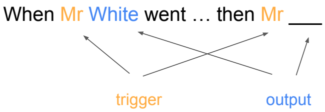
Fix **trigger tokens**: $q_1, \ldots, q_K$

Sample each sequence $z_{1:T} \in [N]^T$ as follows

- **Output tokens**: $o_k \sim \pi_o(\cdot|q_k)$ (*random*)

The bigram data model for in-context reasoning

Goal: capture both in-context and global knowledge (*e.g.*, nouns vs syntax)



When Mr White went to the mall, it started raining, then Mr White witnessed an odd occurrence. While walking around the mall with his family, Mr White heard the sound of a helicopter landing in the parking lot. Curious, he made his way over to see what was going on.

Fix **trigger tokens**: $q_1, \dots, q_K$

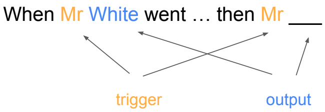
Sample each sequence $z_{1:T} \in [N]^T$ as follows

- **Output tokens**: $o_k \sim \pi_o(\cdot | q_k)$ (*random*)
- **Sequence-specific Markov model**: $z_1 \sim \pi_1$, $z_t | z_{t-1} \sim p(\cdot | z_{t-1})$ with

$$p(j|i) = \begin{cases} \mathbb{1}\{j = o_k\}, & \text{if } i = q_k, \quad k = 1, \dots, K \\ \pi_b(j|i), & \text{o/w.} \end{cases}$$

The bigram data model for in-context reasoning

Goal: capture both in-context and global knowledge (*e.g.*, nouns vs syntax)



When Mr White went to the mall, it started raining, then Mr White witnessed an odd occurrence. While walking around the mall with his family, Mr White heard the sound of a helicopter landing in the parking lot. Curious, he made his way over to see what was going on.

Fix **trigger tokens**: $q_1, \dots, q_K$

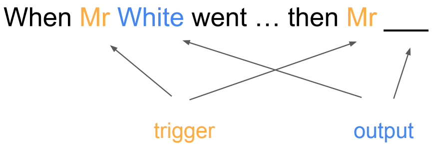
Sample each sequence $z_{1:T} \in [N]^T$ as follows

- **Output tokens**: $o_k \sim \pi_o(\cdot | q_k)$ (*random*)
- **Sequence-specific Markov model**: $z_1 \sim \pi_1$, $z_t | z_{t-1} \sim p(\cdot | z_{t-1})$ with

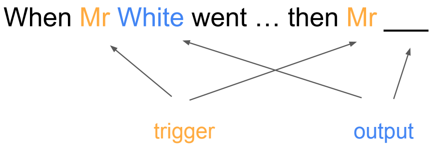
$$p(j|i) = \begin{cases} \mathbb{1}\{j = o_k\}, & \text{if } i = q_k, \quad k = 1, \dots, K \\ \pi_b(j|i), & \text{o/w.} \end{cases}$$

π_b: **global bigrams** model (estimated from Karpathy's character-level Shakespeare)

Transformers on the bigram task

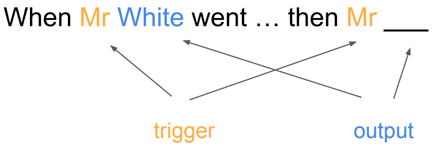


Transformers on the bigram task



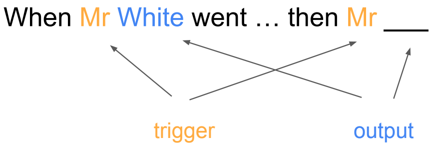
- **1-layer transformer fails**: $\sim 55\%$ accuracy on in-context output predictions

Transformers on the bigram task



- **1-layer transformer fails**: $\sim$ 55% accuracy on in-context output predictions
- **2-layer transformer succeeds**: $\sim$ 99% accuracy

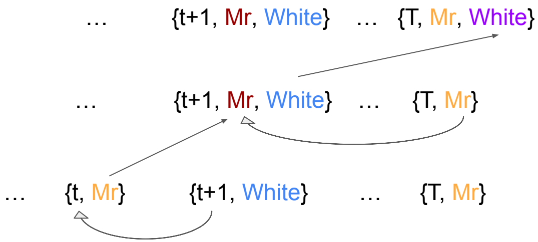
Transformers on the bigram task



- **1-layer transformer fails**: $\sim 55\%$ accuracy on in-context output predictions
- **2-layer transformer succeeds**: $\sim 99\%$ accuracy

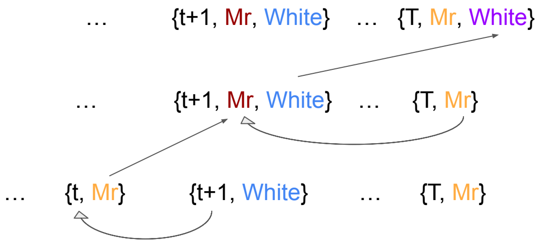
See (Sanford, Hsu, and Telgarsky, 2023, 2024) for representational lower bounds

Induction head mechanism (Elhage et al., 2021; Olsson et al., 2022)



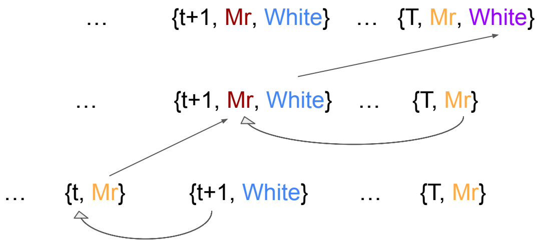
- 1st layer: **previous-token head**
  - attends to previous token and copies it to residual stream

Induction head mechanism (Elhage et al., 2021; Olsson et al., 2022)

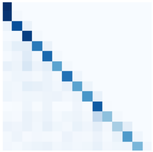


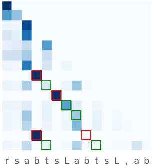
- 1st layer: **previous-token head**
  - attends to previous token and copies it to residual stream
- 2nd layer: **induction head**
  - attends to output of previous token head, copies attended token

Induction head mechanism (Elhage et al., 2021; Olsson et al., 2022)



- 1st layer: **previous-token head**
  - attends to previous token and copies it to residual stream
- 2nd layer: **induction head**
  - attends to output of previous token head, copies attended token
- Matches observed attention scores:





Random embeddings in high dimension

- We consider **random** embeddings u_i with i.i.d. $\mathcal{N}(0, 1/d)$ entries and d large

$$\|u_i\| \approx 1 \quad \text{and} \quad u_i^\top u_j = O(1/\sqrt{d})$$

Random embeddings in high dimension

- We consider **random** embeddings u_i with i.i.d. $\mathcal{N}(0, 1/d)$ entries and d large

$$\|u_i\| \approx 1 \quad \text{and} \quad u_i^\top u_j = O(1/\sqrt{d})$$

- **Remapping**: multiply by random matrix W with $\mathcal{N}(0, 1/d)$ entries:

$$\|Wu_i\| \approx 1 \quad \text{and} \quad u_i^\top W u_i = O(1/\sqrt{d})$$

Random embeddings in high dimension

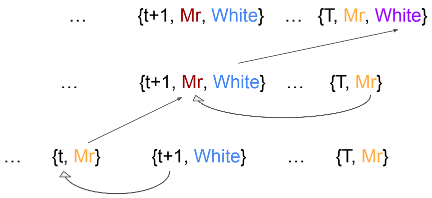
- We consider **random** embeddings u_i with i.i.d. $\mathcal{N}(0, 1/d)$ entries and d large

$$\|u_i\| \approx 1 \quad \text{and} \quad u_i^\top u_j = O(1/\sqrt{d})$$

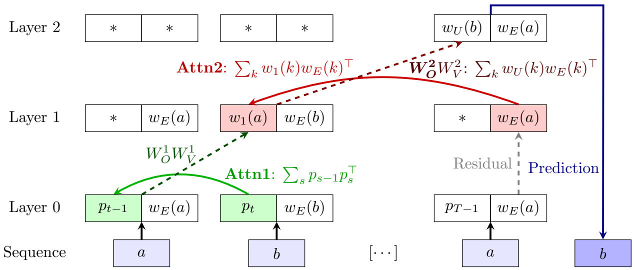
- **Remapping**: multiply by random matrix W with $\mathcal{N}(0, 1/d)$ entries:

$$\|W u_i\| \approx 1 \quad \text{and} \quad u_i^\top W u_i = O(1/\sqrt{d})$$

- Value/Output matrices help with token **remapping**: Mr $\mapsto$ Mr, White $\mapsto$ White



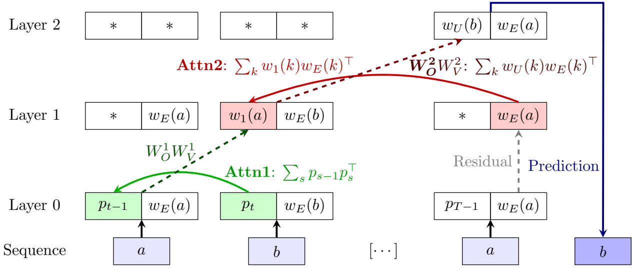
Induction head with associative memories



$$W_{KQ}^1 = \sum_{t=2}^{T} p_t p_{t-1}^\top, \quad W_{KQ}^2 = \sum_{k \in Q} e_k \tilde{e}_k^\top, \quad W_{OV}^2 = \sum_{k=1}^{N} u_k e_k^\top,$$

- Random embeddings e_k, u_k, random matrix W_{OV}^1 (frozen at init)
- **Remapped** previous tokens: $\tilde{e}_k := W_{OV}^1 e_k$

Induction head with associative memories



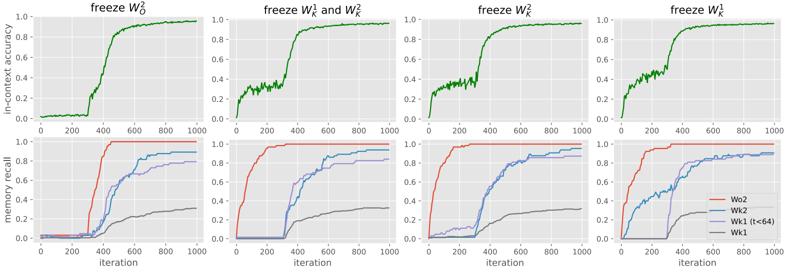
$$W_{KQ}^1 = \sum_{t=2}^{T} p_t p_{t-1}^\top, \quad W_{KQ}^2 = \sum_{k \in Q} e_k \tilde{e}_k^\top, \quad W_{OV}^2 = \sum_{k=1}^{N} u_k e_k^\top,$$

- Random embeddings e_k, u_k, random matrix W_{OV}^1 (frozen at init)
- **Remapped** previous tokens: $\tilde{e}_k := W_{OV}^1 e_k$

Q: Does this match practice?

Empirically probing the dynamics

Train only W_{KQ}^1, W_{KQ}^2, W_{OV}^2, loss on deterministic output tokens only

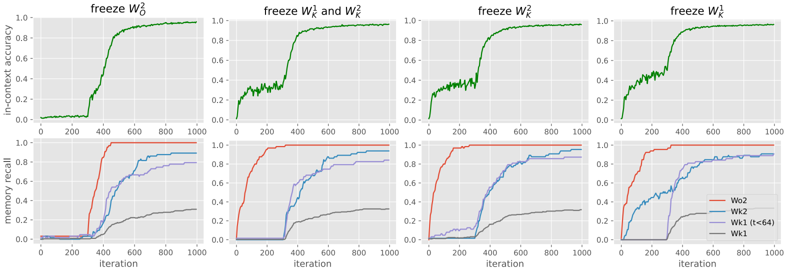


- "Memory recall **probes**": for target memory $W_* = \sum_{i=1}^M u_i e_i^\top$, compute

$$R(\hat{W}, W_*) = \frac{1}{M} \sum_{i=1}^{M} \mathbb{1}\{i = \arg\max_j u_j^\top \hat{W} e_i\}$$

Empirically probing the dynamics

Train only W^1_{KQ}, W^2_{KQ}, W^2_{OV}, loss on deterministic output tokens only



- "Memory recall **probes**": for target memory $W_* = \sum_{i=1}^M u_i e_i^\top$, compute

$$R(\hat{W}, W_*) = \frac{1}{M} \sum_{i=1}^{M} \mathbb{1}\{i = \arg\max_j u_j^\top \hat{W} e_i\}$$

- Natural learning "**order**": W^2_{OV} first, W^2_{KQ} next, W^1_{KQ} last
- Joint learning is faster

Gradient steps for the bigram task

Setting: transformer on the bigram task

- Focus on predicting second output token
- All distributions are uniform
- Some simplifications to architecture
- Infinite width, infinite data, $N \gg T$

Gradient steps for the bigram task

Setting: transformer on the bigram task

- Focus on predicting second output token
- All distributions are uniform
- Some simplifications to architecture
- Infinite width, infinite data, $N \gg T$

Theorem (B. et al., 2023, informal)

In the setup above, we can recover the desired associative memories with ***3 gradient steps*** *on the population loss: first on* W_{OV}^2*, then* W_{KQ}^2*, then* W_{KQ}^1*.*

Gradient steps for the bigram task

Setting: transformer on the bigram task

- Focus on predicting second output token
- All distributions are uniform
- Some simplifications to architecture
- Infinite width, infinite data, $N \gg T$

Theorem (B. et al., 2023, informal)

*In the setup above, we can recover the desired associative memories with **3 gradient steps** on the population loss: first on W_{OV}^2, then W_{KQ}^2, then W_{KQ}^1.*

Key ideas

- Attention is uniform at initialization $\implies$ inputs are sums of embeddings
- W_{OV}^2: correct output appears w.p. 1, while other tokens are noisy and cond. indep. of z_T
- $W_{KQ}^{1/2}$: correct associations lead to more focused attention

Gradient steps for the bigram task

Setting: transformer on the bigram task

- Focus on predicting second output token
- All distributions are uniform
- Some simplifications to architecture
- Infinite width, infinite data, $N \gg T$

Theorem (B. et al., 2023, informal)

*In the setup above, we can recover the desired associative memories with **3 gradient steps** on the population loss: first on W_{OV}^2, then W_{KQ}^2, then W_{KQ}^1.*

Key ideas

- Attention is uniform at initialization $\implies$ inputs are sums of embeddings
- W_{OV}^2: correct output appears w.p. 1, while other tokens are noisy and cond. indep. of z_T
- $W_{KQ}^{1/2}$: correct associations lead to more focused attention

see also (Snell et al., 2021; Oymak et al., 2023)

Key idea: gradient associative memories with noisy inputs

Insight: residual streams, attention output at init, are noisy sums of embeddings

Lemma (Gradients with noisy inputs)

Let p be a data distribution over $(x, y) \in \mathbb{R}^d \times [N]$, and consider the loss

$$L(W) = \mathbb{E}_{(x,y)\sim p}[\ell(y, F_W(x))], \quad F_W(z)_k = u_k{}^\top W x.$$

Key idea: gradient associative memories with noisy inputs

Insight: residual streams, attention output at init, are noisy sums of embeddings

Lemma (Gradients with noisy inputs)

Let p be a data distribution over $(x, y) \in \mathbb{R}^d \times [N]$, *and consider the loss*

$$L(W) = \mathbb{E}_{(x,y)\sim p}[\ell(y, F_W(x))], \quad F_W(z)_k = u_k^\top W x.$$

Denoting $\mu_k := \mathbb{E}[x|y=k]$ *and* $\hat{\mu}_k := \mathbb{E}_x[\frac{\hat{p}_W(k|x)}{p(y=k)} x]$, *we have*

$$\nabla_W L(W) = \sum_{k=1}^N p(y=k) u_k (\hat{\mu}_k - \mu_k)^\top.$$

Key idea: gradient associative memories with noisy inputs

Insight: residual streams, attention output at init, are noisy sums of embeddings

Lemma (Gradients with noisy inputs)

Let p be a data distribution over $(x, y) \in \mathbb{R}^d \times [N]$, and consider the loss

$$L(W) = \mathbb{E}_{(x,y)\sim p}[\ell(y, F_W(x))], \quad F_W(z)_k = u_k{}^\top W x.$$

Denoting $\mu_k := \mathbb{E}[x|y = k]$ and $\hat{\mu}_k := \mathbb{E}_x[\frac{\hat{p}_W(k|x)}{p(y=k)} x]$, we have

$$\nabla_W L(W) = \sum_{k=1}^{N} p(y = k) u_k (\hat{\mu}_k - \mu_k)^\top.$$

- **Example**: $y \sim \text{Unif}([N])$, $t \sim \text{Unif}([T])$, $x = e_y + p_t$.

Key idea: gradient associative memories with noisy inputs

Insight: residual streams, attention output at init, are noisy sums of embeddings

Lemma (Gradients with noisy inputs)

Let p be a data distribution over $(x, y) \in \mathbb{R}^d \times [N]$, and consider the loss

$$L(W) = \mathbb{E}_{(x,y)\sim p}[\ell(y, F_W(x))], \quad F_W(z)_k = u_k{}^\top W x.$$

Denoting $\mu_k := \mathbb{E}[x|y = k]$ and $\hat{\mu}_k := \mathbb{E}_x[\frac{\hat{p}_W(k|x)}{p(y=k)} x]$, we have

$$\nabla_W L(W) = \sum_{k=1}^{N} p(y = k) u_k (\hat{\mu}_k - \mu_k)^\top.$$

- **Example**: $y \sim \text{Unif}([N])$, $t \sim \text{Unif}([T])$, $x = e_y + p_t$. One gradient step:

$$u_k^\top W_1(e_y + p_t) \approx \frac{\eta}{N} \mathbb{1}\{y = k\} + O\left(\frac{1}{N^2}\right)$$

Key idea: gradient associative memories with noisy inputs

Insight: residual streams, attention output at init, are noisy sums of embeddings

Lemma (Gradients with noisy inputs)

Let p be a data distribution over $(x, y) \in \mathbb{R}^d \times [N]$, and consider the loss

$$L(W) = \mathbb{E}_{(x,y)\sim p}[\ell(y, F_W(x))], \quad F_W(z)_k = {u_k}^\top W x.$$

Denoting $\mu_k := \mathbb{E}[x|y = k]$ *and* $\hat{\mu}_k := \mathbb{E}_x[\frac{\hat{p}_W(k|x)}{p(y=k)} x]$, *we have*

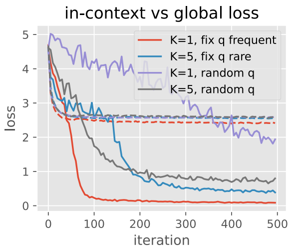
$$\nabla_W L(W) = \sum_{k=1}^N p(y = k) u_k (\hat{\mu}_k - \mu_k)^\top.$$

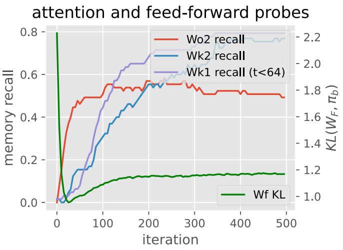
- **Example**: $y \sim \text{Unif}([N])$, $t \sim \text{Unif}([T])$, $x = e_y + p_t$. One gradient step:

$$u_k^\top W_1(e_y + p_t) \approx \frac{\eta}{N} \mathbb{1}\{y = k\} + O\left(\frac{1}{N^2}\right)$$

- Similar arguments for attention matrices

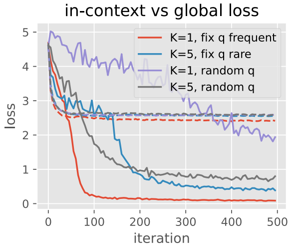
Global vs in-context associations

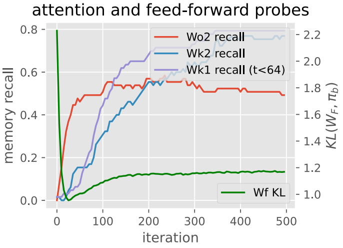


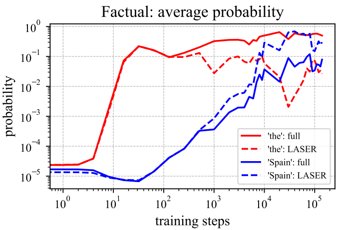


- Global bigrams are learned much faster than induction head, tend to be stored in MLPs

Global vs in-context associations





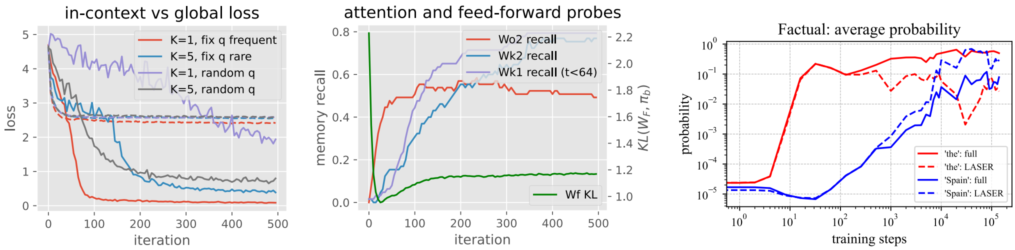


- Global bigrams are learned much faster than induction head, tend to be stored in MLPs

Trade-offs between global and in-context predictions (Chen, Bruna, and B., 2024)

- Trade-offs also appear in LLMs
  - "Madrid is located in" $\rightarrow$ {the, Spain} on Pythia-1B
  - Ablating late-layer MLPs (Sharma et al., 2023) changes prediction from global to in-context

Global vs in-context associations



- Global bigrams are learned much faster than induction head, tend to be stored in MLPs

Trade-offs between global and in-context predictions (Chen, Bruna, and B., 2024)

- Trade-offs also appear in LLMs
  - "Madrid is located in" $\rightarrow$ {the, Spain} on Pythia-1B
  - Ablating late-layer MLPs (Sharma et al., 2023) changes prediction from global to in-context

Theorem (Chen et al., 2024, informal)

In toy setting, feed-forward layer learns global bigram after $O(1)$ samples, attention after $O(N)$ samples due to noise.

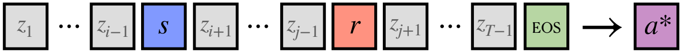
Outline

1 Associative memories

2 Application to Transformers I: reasoning (B. et al., 2023)

3 Application to Transformers II: factual recall (Nichani et al., 2024)

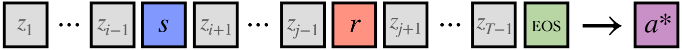
Toy model of factual recall



The capital of France is Paris

- $s \in \mathcal{S}$: subject token
- $r \in \mathcal{R}$: relation token
- $a^*(s, r) \in \mathcal{A}_r$: attribute/fact to be stored
- $z_i \in \mathcal{N}$: noise tokens

Toy model of factual recall



The capital of France is Paris

- $s \in \mathcal{S}$: subject token
- $r \in \mathcal{R}$: relation token
- $a^*(s, r) \in \mathcal{A}_r$: attribute/fact to be stored
- $z_i \in \mathcal{N}$: noise tokens

Q: How many parameters do Transformers need to solve this?

How many parameters do we need?

- One-layer Transformer, with or without MLP, random embeddings
- Embedding dimension d, head dimension d_h, MLP width m, H heads

Theorem (Nichani et al., 2024, informal)

- *Attention + MLP:* $Hd_h \gtrsim S + R$ *and* $md \gtrsim SR$ *succeeds*
- *Attention-only:* $d \gtrsim R + A_{\max}$ *and* $Hd_h \gtrsim S$ *succeeds* ($A_{\max} := \max_r |\mathcal{A}_r|$)

How many parameters do we need?

- One-layer Transformer, with or without MLP, random embeddings
- Embedding dimension d, head dimension d_h, MLP width m, H heads

Theorem (Nichani et al., 2024, informal)

- *Attention + MLP:* $Hd_h \gtrsim S + R$ *and* $md \gtrsim SR$ *succeeds*
- *Attention-only:* $d \gtrsim R + A_{\max}$ *and* $Hd_h \gtrsim S$ *succeeds* ($A_{\max} := \max_r |\mathcal{A}_r|$)

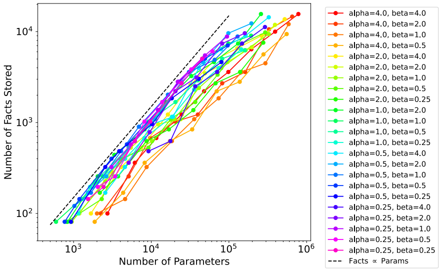
- Total parameters scale with number of facts SR (up to $A_{\max}$)
- Constructions are based on associative memories
- Attention-only needs large enough d
- Noise is negligible (log factors)

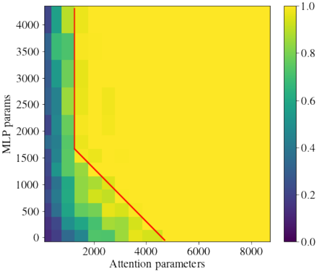
How many parameters do we need?

- One-layer Transformer, with or without MLP, random embeddings
- Embedding dimension d, head dimension d_h, MLP width m, H heads

Theorem (Nichani et al., 2024, informal)

- *Attention + MLP:* $Hd_h \gtrsim S + R$ *and* $md \gtrsim SR$ *succeeds*
- *Attention-only:* $d \gtrsim R + A_{\max}$ *and* $Hd_h \gtrsim S$ *succeeds* ($A_{\max} := \max_r |\mathcal{A}_r|$)





Training dynamics

- One-layer Transformer with **linear attention** and **one-hot** embeddings
- Gradient flow with initialization $W_{OV}(a, z), w_{KQ}(z) \approx \alpha > 0$

Theorem (Nichani et al., 2024, informal)

- *We have global convergence to zero loss*
- *There is an intermediate phase where the model predicts with $p(a|r)$ instead of $p(a|s, r)$*

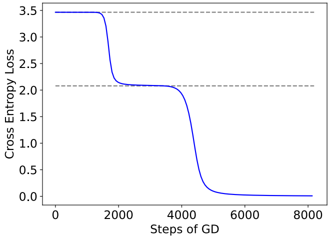
Training dynamics

- One-layer Transformer with **linear attention** and **one-hot** embeddings
- Gradient flow with initialization $W_{OV}(a, z), w_{KQ}(z) \approx \alpha > 0$

Theorem (Nichani et al., 2024, informal)

- *We have global convergence to zero loss*
- *There is an intermediate phase where the model predicts with $p(a|r)$ instead of $p(a|s, r)$*

- Intermediate phase corresponds to **hallucination** (over $\mathcal{A}_r$, ignoring s)



Concluding remarks

Transformer weights as associative memories

- Storage capacity and gradient-based learning
- Toy models of reasoning and factual recall
- Linear models <3

Concluding remarks

Transformer weights as associative memories

- Storage capacity and gradient-based learning
- Toy models of reasoning and factual recall
- Linear models <3

Extensions (Cabannes et al., 2024a,b)

- Scaling laws for power law data (finite d and n)
- Optimization beyond one step

Concluding remarks

Transformer weights as associative memories

- Storage capacity and gradient-based learning
- Toy models of reasoning and factual recall
- Linear models <3

Extensions (Cabannes et al., 2024a,b)

- Scaling laws for power law data (finite d and n)
- Optimization beyond one step

Future directions

- Finite sample results
- More complex reasoning problems
- Fine-grained optimization
- Learning embeddings

Concluding remarks

Transformer weights as associative memories

- Storage capacity and gradient-based learning
- Toy models of reasoning and factual recall
- Linear models <3

Extensions (Cabannes et al., 2024a,b)

- Scaling laws for power law data (finite d and n)
- Optimization beyond one step

Future directions

- Finite sample results
- More complex reasoning problems
- Fine-grained optimization
- Learning embeddings

Thank you!

References I

Z. Allen-Zhu and Y. Li. Physics of language models: Part 3.3, knowledge capacity scaling laws. *arXiv preprint arXiv:2404.05405*, 2024.

A. B., V. Cabannes, D. Bouchacourt, H. Jegou, and L. Bottou. Birth of a transformer: A memory viewpoint. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2023.

J. Ba, M. A. Erdogdu, T. Suzuki, Z. Wang, D. Wu, and G. Yang. High-dimensional asymptotics of feature learning: How one gradient step improves the representation. *Advances in Neural Information Processing Systems (NeurIPS)*, 2022.

V. Cabannes, E. Dohmatob, and A. B. Scaling laws for associative memories. In *International Conference on Learning Representations (ICLR)*, 2024a.

V. Cabannes, B. Simsek, and A. B. Learning associative memories with gradient descent. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2024b.

L. Chen, J. Bruna, and A. B. How truncating weights improves reasoning in language models. *arXiv preprint arXiv:2406.03068*, 2024.

A. Damian, J. Lee, and M. Soltanolkotabi. Neural networks can learn representations with gradient descent. In *Conference on Learning Theory (COLT)*, 2022.

M. Demircigil, J. Heusel, M. Löwe, S. Upgang, and F. Vermet. On a model of associative memory with huge storage capacity. *Journal of Statistical Physics*, 168:288–299, 2017.

References II

N. Elhage, N. Nanda, C. Olsson, T. Henighan, N. Joseph, B. Mann, A. Askell, Y. Bai, A. Chen, T. Conerly, N. DasSarma, D. Drain, D. Ganguli, Z. Hatfield-Dodds, D. Hernandez, A. Jones, J. Kernion, L. Lovitt, K. Ndousse, D. Amodei, T. Brown, J. Clark, J. Kaplan, S. McCandlish, and C. Olah. A mathematical framework for transformer circuits. *Transformer Circuits Thread*, 2021.

M. Geva, R. Schuster, J. Berant, and O. Levy. Transformer feed-forward layers are key-value memories. *arXiv preprint arXiv:2012.14913*, 2020.

J. J. Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the national academy of sciences*, 79(8):2554–2558, 1982.

A. Iscen, T. Furon, V. Gripon, M. Rabbat, and H. Jégou. Memory vectors for similarity search in high-dimensional spaces. *IEEE transactions on big data*, 4(1):65–77, 2017.

T. Kohonen. Correlation matrix memories. *IEEE Transactions on Computers*, 1972.

D. Krotov and J. J. Hopfield. Dense associative memory for pattern recognition. *Advances in neural information processing systems*, 29, 2016.

B. Liu, J. T. Ash, S. Goel, A. Krishnamurthy, and C. Zhang. Transformers learn shortcuts to automata. In *Proceedings of the International Conference on Learning Representations (ICLR)*, 2023.

K. Meng, D. Bau, A. Andonian, and Y. Belinkov. Locating and editing factual associations in GPT. *Advances in Neural Information Processing Systems (NeurIPS)*, 2022.

References III

W. Merrill, A. Sabharwal, and N. A. Smith. Saturated transformers are constant-depth threshold circuits. *Transactions of the Association for Computational Linguistics*, 10:843–856, 2022.

E. Nichani, J. D. Lee, and A. B. Understanding factual recall in transformers via associative memories. *arXiv preprint arXiv:2412.06538*, 2024.

C. Olsson, N. Elhage, N. Nanda, N. Joseph, N. DasSarma, T. Henighan, B. Mann, A. Askell, Y. Bai, A. Chen, T. Conerly, D. Drain, D. Ganguli, Z. Hatfield-Dodds, D. Hernandez, S. Johnston, A. Jones, J. Kernion, L. Lovitt, K. Ndousse, D. Amodei, T. Brown, J. Clark, J. Kaplan, S. McCandlish, and C. Olah. In-context learning and induction heads. *Transformer Circuits Thread*, 2022.

S. Oymak, A. S. Rawat, M. Soltanolkotabi, and C. Thrampoulidis. On the role of attention in prompt-tuning. In *International Conference on Machine Learning*, 2023.

H. Ramsauer, B. Schäfl, J. Lehner, P. Seidl, M. Widrich, T. Adler, L. Gruber, M. Holzleitner, M. Pavlović, G. K. Sandve, et al. Hopfield networks is all you need. *arXiv preprint arXiv:2008.02217*, 2020.

C. Sanford, D. Hsu, and M. Telgarsky. Representational strengths and limitations of transformers. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2023.

C. Sanford, D. Hsu, and M. Telgarsky. One-layer transformers fail to solve the induction heads task. *arXiv preprint arXiv:2408.14332*, 2024.

References IV

I. Schlag, K. Irie, and J. Schmidhuber. Linear transformers are secretly fast weight programmers. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2021.

P. Sharma, J. T. Ash, and D. Misra. The truth is in there: Improving reasoning in language models with layer-selective rank reduction. *arXiv preprint arXiv:2312.13558*, 2023.

C. Snell, R. Zhong, D. Klein, and J. Steinhardt. Approximating how single head attention learns. *arXiv preprint arXiv:2103.07601*, 2021.

K. Wang, A. Variengien, A. Conmy, B. Shlegeris, and J. Steinhardt. Interpretability in the wild: a circuit for indirect object identification in gpt-2 small. *arXiv preprint arXiv:2211.00593*, 2022.

D. J. Willshaw, O. P. Buneman, and H. C. Longuet-Higgins. Non-holographic associative memory. *Nature*, 222(5197):960–962, 1969.

G. Yang and E. J. Hu. Tensor programs iv: Feature learning in infinite-width neural networks. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2021.